

A SOLUTION TO SCHROEDER'S EQUATION IN SEVERAL VARIABLES

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ABSTRACT. Let ϕ be a self-map of \mathbb{B}^n , the unit ball in \mathbb{C}^n , fixing 0, and having full-rank at 0. If $\phi'(0) \neq 0$, Koenigs proved in 1884 that in the well-known case $n = 1$, Schroeder's equation, $f \circ \phi = \lambda f$ has a solution f , which is bijective near 0 precisely when $\lambda = \phi'(0)$. In 2003, Cowen and MacCluer formulated the analogous problem in \mathbb{C}^n (for a non-negative integer n) by defining Schroeder's equation in several variables as $F \circ \phi = \phi'(0)F$ and giving appropriate assumptions on ϕ . The 2003 Cowen and MacCluer paper also provides necessary and sufficient conditions for an analytic solution, F taking values in \mathbb{C}^n and having full-rank near 0 under the additional assumption that $\phi'(0)$ is diagonalizable. The main result of this paper gives necessary and sufficient conditions for a Schroeder solution F which has full rank near 0 without the added assumption of diagonalizability. More generally, it is proven in this paper that the functional equation $C_\phi F = \phi'(0)^k F$ with k a positive integer, is always solvable with an F whose component functions are linearly independent, but if $k > 1$ any such F cannot be injective near 0. In 2007 Enoch provides many theorems giving formal power series solutions to Schroeder's equation in several variables. It is also proved in this note that any formal power series solution indeed represents an analytic function on the ball.

1. INTRODUCTION

Let ϕ be a self-map of $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, fixing 0, and not a disk automorphism. To avoid trivialities, assume that ϕ is not the zero function. Schroeder's equation is $f \circ \phi = \lambda f$, where f is an unknown

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analytic function on \mathcal{D} , and λ an unknown complex number. If C_ϕ denotes the composition operator, which sends a function f defined on \mathbb{D} to $f \circ \phi$, then Schroeder's equation is the eigenvalue equation, $C_\phi f = \lambda f$. Koenigs showed in 1884 that Schroeder's equation has a solution, f , which is also bijective near 0, if and only if $\lambda = \phi'(0) \neq 0$ [Koe84]. Now if $f'(0) \neq 0$, we can view Schroeder's equation not only as an eigenvalue equation, but as a change of variables near 0, the attracting fixed point of ϕ . Explicitly, $\phi(z) = f^{-1}(\phi'(0)z)$ near 0. Such a solution was one of the first steps in understanding intertwining maps and models of iteration, a theory which has been foundational for the understanding of composition operators in one variable (for example, see [CM95, Section 2.4]).

In hopes of generalization, Cowen and MacCluer have formulated an analogous problem in \mathbb{C}^n (for arbitrary n), namely to solve the functional equation

$$(1) \quad C_\phi F = \phi'(0)F$$

with an analytic $F : \mathbb{B}^n \rightarrow \mathbb{C}^n$, where ϕ is a given analytic self-map of \mathbb{B}^n , the unit ball in \mathbb{C}^n [CM03]. The hypotheses taken on ϕ are that $\phi(0) = 0$, $\phi'(0)$ has full rank, and ϕ is not unitary on any slice. Notice that the statement “ $|\phi(z)| < |z|$ for any $0 < |z| < 1$ ” is equivalent to “ ϕ is not unitary on a slice” via Schwarz's lemma. Since ϕ is \mathbb{C}^n -valued, we see that $\phi'(0)$ is an $n \times n$ matrix, so F must be a column vector with n components, that is,

$$F = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} = (f_1, \dots, f_n)^t$$

with each $f_j : \mathbb{B}^n \rightarrow \mathbb{C}$ analytic so the multiplication on the right side of (1) makes sense. For $n > 1$ Equation (1) is no longer an eigenvalue equation but is still a change of variables near 0, provided F has full rank near 0. As in the one variable setting, $\phi(z) = F^{-1}(\phi'(0)F(z))$ near 0. Thus, we seek solutions $F : \mathbb{B}^n \rightarrow \mathbb{C}^n$ which are analytic and have full rank near 0, and any such F will be referred to as a “full rank solution.” For more details on the formation of the problem, see [CM03]. Unlike the single variable case, it is easy to see that if $n > 1$ there exist Schroeder solutions which are nontrivial and not full rank either, for example if $\phi(z_1, z_2) = (z_1/2, z_2/4)$ and $F(z_1, z_2) = (z_1, z_1^2)^t$.

After formulating Schroeder's equation in several variables, [CM03] gives necessary and sufficient conditions for a full rank solution under the additional hypothesis that $\phi'(0)$ is diagonalizable. Specifically, their argument exhibits a Bergman space of analytic function on \mathbb{B}^n which is

large enough so that C_ϕ is compact on it, and then uses the compactness to produce solutions $F = (f_1, \dots, f_n)^t$ with each f_j in the Bergman space. This ensures that F is indeed analytic. The current article adopts the same overall strategy but provides a new viewpoint, namely reducing to the case that $\phi'(0)$ is in Jordan form. It is this insight that allows us to give necessary and sufficient conditions for a full rank solution under the general hypotheses, that is, without the additional assumption that $\phi'(0)$ is diagonalizable (see Theorem 6.3 and 6.4).

It is well-known in the one variable setting that $\phi'(0)^k$ is an eigenvalue of C_ϕ for any positive integer k . We consider the analogous equation,

$$(2) \quad C_\phi F = \phi'(0)^k F$$

with k any positive integer, and show under the same hypotheses on ϕ that a solution F with linearly independent component functions always exists (see Theorem 7.2).

1.1. Outline. Sections 2 and 3 will provide the details of reducing to $\phi'(0)$ in Jordan form, introduce necessary notation, and see what is gained by such a simplification. Reducing to Jordan form is (to the author's knowledge) not previously considered in the literature, and this new point of view is the "trick" to understanding solutions. Section 4 uses the compactness of C_ϕ to reduce the problem of finding Schroeder solutions to that of linear algebra, and contains a subsection of the necessary linear algebra results. It is here that much of the work takes place. The main result of this section will guarantee the existence of a Schroeder solution, F with linearly independent component functions. This gives an intermediate solution (between no solution and a full rank solution) when no full rank solution exists. Section 5 examines obstructions to a full rank solution, and the role resonant eigenvalues play; specifically, the Jordan form viewpoint allows one to see specifically how a resonant eigenvalue prevents the existence of a solution which is locally univalent near 0. Our main result lies in Section 6, which tackles the problem of discriminating which ϕ admit full rank solutions to Schroeder's equation, and the closing section addresses solutions to (2).

2. SIMPLIFYING ASSUMPTIONS

Let ϕ will denote an analytic self-map of \mathbb{B}^n , fixing 0. Let D be an invertible $n \times n$ matrix such that $D\phi'(0)D^{-1}$ is in Jordan form. Notice $D\mathbb{B}^n$ is an ellipsoid, so while $D\phi D^{-1}$ does have a Jordan form derivative at 0, it is not necessarily defined on \mathbb{B}^n . This á priori eliminates using $C_{D\phi D^{-1}}$ on a Hilbert space of analytic functions on \mathbb{B}^n . To continue

we will construct a Hilbert space of analytic functions on $D\mathbb{B}^n$, so that $C_{D\phi D^{-1}}$ is not only defined on this space, but is even a compact operator.

Recall that given a domain \mathcal{D} of \mathbb{C}^n and $G : \mathcal{D} \rightarrow (0, \infty)$, $A_G^2(\mathcal{D})$ denotes the space of analytic function $f : \mathcal{D} \rightarrow \mathbb{C}$ satisfying

$$\|f\|^2 := \int_{\mathcal{D}} |f(z)|^2 G(z) dz < \infty$$

where dz denotes Lebesgue measure on \mathbb{R}^{2n} . When the domain $\mathcal{D} = \mathbb{B}^n$ we may simply write A_G^2 .

Clearly $f \mapsto f \circ D^{-1}$ is a bijective map of $\mathcal{O}(\mathbb{B}^n)$ onto $\mathcal{O}(D\mathbb{B}^n)$, where $\mathcal{O}(\mathcal{D})$ denotes the analytic functions on a domain \mathcal{D} . Since,

$$\int_{\mathbb{B}^n} |f|^2(z) G(z) dz = \int_{D\mathbb{B}^n} |f \circ (D^{-1})|^2(w) \frac{G \circ (D^{-1})(w)}{|\det(D)|^2} dw,$$

we see $A_H^2(D\mathbb{B}^n)$ is isometrically isomorphic to A_G^2 for

$$H = \frac{G \circ D^{-1}}{|\det(D)|^2}.$$

This gives rise to the following proposition.

Proposition 2.1. *With ϕ and H as above, the following are true.*

- (1) *There exists D , an invertible $n \times n$ matrix such that $D\phi'(0)D^{-1}$ is in Jordan form.*
- (2) *A_G^2 is isometrically isomorphic to $A_H^2(D\mathbb{B}^n)$ via $\iota(f) = f \circ D^{-1}$.*
- (3) *C_ϕ is bounded (compact) on A_G^2 if and only if $C_{D\phi D^{-1}}$ is bounded (compact) on $A_H^2(D\mathbb{B}^n)$.*
- (4) *$\{z^\alpha\}_\alpha$ for multi-indices $\alpha \in (\mathbb{N} \cup \{0\})^N$ is a linearly independent subset of both A_G^2 and $A_H^2(D\mathbb{B}^n)$ (although not necessarily orthogonal in $A_H^2(D\mathbb{B}^n)$).*

In practice we usually represent functions of $A_H^2(D\mathbb{B}^n)$ using their Taylor series. (4) is stated as a reminder that the monomials are not orthogonal vectors in $A_H^2(D\mathbb{B}^n)$. Although, with the aid of Proposition 2.2 we see that the subsequence of homogeneous polynomials converges in $A_H^2(D\mathbb{B}^n)$, and so this convenient representation makes sense in $A_H^2(D\mathbb{B}^n)$.

Proof. All statements up to and including the isometric isomorphism have been proven. For (3), noting that the following diagram commutes,

$$\begin{array}{ccc} A_G^2 & \xrightarrow{C_\phi} & A_G^2 \\ \downarrow \iota & & \downarrow \iota \\ A_H^2(D\mathbb{B}^n) & \xrightarrow{C_{D\phi D^{-1}}} & A_H^2(D\mathbb{B}^n) \end{array}$$

we see C_ϕ is bounded (compact) on A_G^2 if and only if $C_{D\phi D^{-1}}$ is bounded (compact) on $A_H^2(D\mathbb{B}^n)$.

For (4) it is well known that $\{z^\alpha\}$ is orthogonal (and therefore linearly independent) in A_G^2 . For the last result, assume

$$\sum_{j=1}^n c_j z^{\alpha_j} = 0$$

in $A_H^2(D\mathbb{B}^n)$ for constants c_j . This means that the polynomial on the left is 0 for every z in the open set $D\mathbb{B}^n$, which yields $c_j = 0$ for each j . Thus we have shown the linear independence of $\{z^\alpha\}$.

To see orthogonality of $\{z^\alpha\}$ can fail in $A_H^2(D\mathbb{B}^n)$, consider $G \equiv 1$ on \mathbb{B}^2 , and

$$D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

A simple calculation shows

$$\begin{aligned} \langle z_1, z_2 \rangle_{A_H^2(D\mathbb{B}^2)} &= \int_{D\mathbb{B}^2} z_1 \bar{z}_2 \frac{dz}{|\det(D)|^2} \\ &= \int_{\mathbb{B}^2} (z_1 + z_2) \bar{z}_2 dz \\ &= \langle z_2, z_2 \rangle_{A_G^2} > 0. \end{aligned}$$

□

Proposition 2.2. *Suppose $f = \sum_{j=0}^\infty f_j$ is a member of A_G^2 with f_j homogeneous polynomials of degree j . Setting $g = \iota f$ and $g_j = \iota f_j$, we see*

- (1) g_j is also a homogeneous polynomial of degree j ,
- (2) in $A_H^2(D\mathbb{B}^n)$ $\langle g_j, g_k \rangle = 0$ for $k \neq j$, and
- (3) $\sum g_j \rightarrow g$ in $A_H^2(D\mathbb{B}^n)$.

Proof. For any λ in \mathbb{B}^n , we have

$$\begin{aligned} g_j(\lambda z) &= \iota f_j(\lambda z) \\ &= f_j D^{-1}(\lambda z) \\ &= f_j(\lambda D^{-1}z) \\ &= \lambda^j f_j D^{-1}(z) = \lambda^j g_n(z) \end{aligned}$$

which shows (1). Both (2) and (3) follow immediately from (1) and the fact ι is an isometric isomorphism. \square

Lemma 2.3. *With ϕ and D as above, F is a Schroeder solution for $C_{D\phi D^{-1}}$ (with domain $D\mathbb{B}^n$) if and only if $D^{-1}FD$ is a Schroeder solution for C_ϕ (with domain \mathbb{B}^n).*

Proof. Suppose F is a Schroeder solution for $C_{D\phi D^{-1}}$. Clearly, $D^{-1}FD$ has full rank near 0 since F does, and D invertible. So we only need to show

$$\begin{aligned} C_\phi(D^{-1}FD) &= \phi'(0)D^{-1}FD \\ \iff D^{-1}F(D\phi(z)) &= \phi'(0)D^{-1}F(Dz) \text{ whenever } z \text{ in } \mathbb{B}^n \\ \iff F(D\phi(z)) &= D\phi'(0)D^{-1}F(Dz) \text{ whenever } z \text{ in } \mathbb{B}^n \\ \iff F(D\phi(D^{-1}w)) &= D\phi'(0)D^{-1}F(w) \text{ whenever } w \text{ in } D\mathbb{B}^n \\ \iff C_{D\phi D^{-1}}F &= D\phi'(0)D^{-1}F \end{aligned}$$

so we have the result. \square

Recall that our goal is to find a Schroeder solution under the hypotheses that ϕ is a self-map of \mathbb{B}^n fixing 0, not unitary on a slice, and $\phi'(0)$ invertible. Cowen and MacCluer provide sufficient conditions on G so that C_ϕ is compact on the space A_G^2 , and we state their theorem below. As in [CM03], the compactness of C_ϕ will allow us to produce solutions whose component functions lie in A_G^2 , which ensures the analyticity of F .

Theorem 2.4. [CM03, Theorem 8] *If $\phi : \mathbb{B}^n \rightarrow \mathbb{B}^n$ is analytic, fixing 0, and ϕ is not unitary on any slice, then C_ϕ is Hilbert-Schmidt (and therefore compact) on A_G^2 for any continuous, non-increasing $G : [0, 1) \rightarrow (0, \infty)$ such that*

$$\frac{G(r)}{(1-r)^{2n}G(1-\rho(1-r))}$$

is bounded near 1 for any $\rho > 1$. In particular, C_ϕ is Hilbert-Schmidt on A_G^2 for $G(|z|) = \exp(-q/(1-|z|))$, $q > 0$.

Fix G such that C_ϕ is compact on the Bergman space A_G^2 .

Corollary 2.5. $C_{D\phi D^{-1}}$ is compact on $A_H^2(D\mathbb{B}^n)$.

Theorem 2.3 allows us to show the result for $D\phi D^{-1}$. Thus, by replacing ϕ with $D\phi D^{-1}$ and A_G^2 with $A_H^2(D\mathbb{B}^n)$, we may assume $\phi'(0)$ is in Jordan form and retain the compactness of $C_{D\phi D^{-1}}$. Of course the exchange is not free, and the cost for this simplification is that the domain is now $D\mathbb{B}^n$, and the convenient basis vectors, namely, $\{z^\alpha\}_\alpha$ are no longer orthogonal.

2.1. Why Jordan Form? The upshot of assuming $\phi'(0)$ is in Jordan form is that we can easily reformulate Schroder's equation into a statement about the chains of C_ϕ .

Definition 2.6. Let L be a linear operator on a vector space. We say e_1, \dots, e_k are a chain with eigenvalue λ and length k if (and only if) they are non-zero vectors satisfying

$$(3) \quad (L - \lambda I)e_j = \begin{cases} e_{j-1} & \text{if } 2 \leq j \leq k \\ 0 & \text{if } j = 1. \end{cases}$$

Note that the existence of a chain is equivalent to $[\ker(L - \lambda I)] \cap [(L - \lambda I)^{k-1}(X)]$ being nontrivial and also that the compression of L to $\langle e_1, \dots, e_k \rangle$ is just the upper triangular λ -Jordan block of length k ,

$$\begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}.$$

We also notice that reversing the indices of the chain gives e_k, \dots, e_1 , a chain of the lower triangular λ -Jordan block

$$\begin{bmatrix} \lambda & & & \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{bmatrix}.$$

Consider the following (abstract) example.

Example 2.7. Let

$$\phi'(0) = \begin{bmatrix} \lambda & 1 & & \\ 0 & \lambda & & \\ & & \alpha & 1 & 0 \\ & & 0 & \alpha & 1 \\ & & 0 & 0 & \alpha \end{bmatrix}$$

a Jordan form matrix with 2 Jordan blocks, one of length 2 and eigenvalue λ , and one of length 3 and eigenvalue α . For our problem it will be useful to consider the Jordan blocks as defining chains of $\phi'(0)$. This $\phi'(0)$ has two chains, namely

$$(\phi'(0) - \lambda I_5) : e_2 \mapsto e_1 \mapsto 0,$$

and

$$(\phi'(0) - \alpha I_5) : e_5 \mapsto e_4 \mapsto e_3 \mapsto 0.$$

Now consider solving Schroeder's equation with this ϕ . Writing $F = (f_1, \dots, f_5)^t$ we see

$$\begin{bmatrix} C_\phi f_1 \\ \vdots \\ C_\phi f_5 \end{bmatrix} = \phi'(0) \begin{bmatrix} f_1 \\ \vdots \\ f_5 \end{bmatrix}$$

if and only if

$$(C_\phi - \lambda I) : f_1 \mapsto f_2 \mapsto 0$$

and

$$(C_\phi - \alpha I) : f_3 \mapsto f_4 \mapsto f_5 \mapsto 0.$$

Thus, a solution to Schroeder's equation exists if and only if C_ϕ has a λ -chain of length (at least) 2, and an α -chain of length (at least) 3. Suppose for the moment that C_ϕ has a λ -chain of length strictly greater than 2; for example,

$$(C_\phi - \lambda I) : g_k \mapsto \dots \mapsto g_2 \mapsto g_1 \mapsto 0.$$

Then only the last two elements of the chain can be used to solve Schroeder's equation; that is, $f_2 = g_2$ and $f_1 = g_1$ are the only elements of the chain $\{g_i\}$ that can appear as the first two coordinate functions of F . This idea is made rigorous in Lemma 3.2.

Putting $\phi'(0)$ into Jordan form provides a convenient set-up for the problem. Specifically, it gives a basis of \mathbb{C}^n that is a union of chains of $\phi'(0)$, and we now know that solutions to Schroeder's equation depend on finding the corresponding chains of C_ϕ . The next step will be to use the compactness of C_ϕ to reduce the problem to understanding the chains of a particular finite rank operator. (It is denoted U throughout the paper). Representing this operator as an $N \times N$ matrix, and then putting it into Jordan form will yield its chains. So Jordan form not only simplifies the problem, but it plays a role in the main proofs of the paper.

3. NOTATION

In the spirit of the previous section, we now let ϕ be a self-map of \mathcal{D} , an ellipsoid (that is, $\mathcal{D} = D\mathbb{B}^n$ for some invertible matrix D) fixing 0, and having $\phi'(0)$ an invertible, upper-triangular, Jordan-form matrix. Further, we suppose the composition operator C_ϕ is compact on $A_H^2(\mathcal{D})$. For convenience we will often represent elements g of $A_H^2(\mathcal{D})$ as (infinite) column vectors and C_ϕ as an (infinite) matrix. To do this, given such a g , we know $g = \sum_\alpha a_\alpha z^\alpha$ with absolute convergence on \mathcal{D} . Thus, we write

$$g = \begin{bmatrix} a_{(0,0,\dots,0)} \\ a_{(1,0,\dots,0)} \\ a_{(0,1,\dots,0)} \\ \vdots \end{bmatrix}.$$

It is important to note that since we have proven only that $\sum_{j=0}^n g_j \rightarrow g$ in $A_H^2(\mathcal{D})$ with g_j homogeneous polynomials of degree j , we understand $g = \sum_\alpha a_\alpha z^\alpha$ as the sub-sequential limit of homogeneous polynomials when dealing with convergence in the norm of $A_H^2(\mathcal{D})$.

We assume the dictionary ordering on the multi-indices; specifically, $\alpha < \beta$ if and only if either $|\alpha| < |\beta|$, or $|\alpha| = |\beta|$ and writing $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ we also have $\alpha_j = \beta_j$ for $j < j_0$, and $\alpha_{j_0} > \beta_{j_0}$ for some $1 \leq j_0 \leq n$.

Our matrix representation of C_ϕ follows from this ordering of $\{z^\alpha\}_\alpha$. If the j^{th} monomial is z^α , then the j^{th} column of C_ϕ will be the column vector ϕ^α . For example, if we are working in \mathbb{C}^2 and $\phi(z) = (\phi_1, \phi_2)$ with $\phi_1(z) = \lambda_1 z_1 + \lambda_2 z_2 + \sum_{|\alpha|>1} a_\alpha z^\alpha$, and $\phi_2(z) = \zeta_2 z_2 + \sum_{|\alpha|>1} b_\alpha z^\alpha$, we see $C_\phi : 1 \mapsto 1, z_1 \mapsto \phi_1, z_2 \mapsto \phi_2$ and so forth. Thus,

$$C_\phi = \begin{array}{c} 1 \\ z_1 \\ z_2 \\ z_1^2 \\ \vdots \end{array} \begin{bmatrix} 1 & \phi_1 & \phi_2 & \phi_1^2 & \\ 1 & 0 & 0 & 0 & \dots \\ 0 & \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 & \zeta_2 & 0 & \dots \\ 0 & a_{1,0} & b_{1,0} & \lambda_1^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Notice that the first row and column of C_ϕ will always be $(1, 0, \dots)$ or $(1, 0, \dots)^t$, respectively, so we will omit these when representing C_ϕ . In other words, we will consider C_ϕ acting on the subspace of functions which fix 0. With this omission, we see the upper left $n \times n$ corner of C_ϕ is $\phi'(0)^t$. Lastly, we note that $\phi'(0)$ upper triangular implies C_ϕ is lower triangular, which is exemplified above.

Other important observations are that the diagonal entries of C_ϕ , $c_{j,j} = \partial\phi_j(0)/\partial z_j$ for $j = 1, \dots, n$, and in general, $c_{j,j} = (c_{1,1}, \dots, c_{n,n})^\alpha = c_{1,1}^{\alpha_1} \dots c_{n,n}^{\alpha_n}$ where z^α is the j^{th} monomial. This notation follows the literature. A more thorough treatment can be found in [CM03].

Since an upper triangular $\phi'(0)$ implies C_ϕ is lower triangular, when putting $\phi'(0)$ into Jordan form, we will opt for the upper triangular version. As $\phi'(0)^t$ appears in C_ϕ , we can see that lower triangular Jordan blocks will appear on the left side of Schroeder's equation while their transposes, upper triangular Jordan blocks, appear on the right side. For notational ease we agree that J (or often J_i with $i \in \mathbb{N}$) will denote a lower triangular Jordan block throughout the paper, and we will use J^t to denote the upper triangular counterpart. Recall that a Jordan matrix is a matrix with either all upper triangular or all lower triangular Jordan blocks along the diagonal, and all other entries 0. In particular, the Jordan matrix $\phi'(0)$ will be written in block form as

$$\phi'(0) = \begin{bmatrix} J_1^t & & \\ & \ddots & \\ & & J_m^t \end{bmatrix}$$

with each J_j^t an upper triangular Jordan block. We call the subspace on which J_i^t acts, say S_i , a Jordan subspace. This is made rigorous in the definition below.

Definition 3.1. Given a linear operator L on a vector space X , we say S is a Jordan subspace of L with eigenvalue λ if (and only if)

- (1) $X = S \oplus T$ with T an algebraic complement of S .
- (2) $L = \begin{bmatrix} J & 0 \\ 0 & K \end{bmatrix}$, with J and K the compressions of L to S and T , respectively.
- (3) J is a λ -Jordan block.

We will say J has length $k = \dim(S)$.

Since J is lower triangular there is a basis e_1, \dots, e_k for S such that

$$(4) \quad (L - \lambda I)e_j = \begin{cases} e_{j+1} & \text{if } 1 \leq j \leq k-1 \\ 0 & \text{if } j = k, \end{cases}$$

or concisely, e_1, \dots, e_k is a chain basis for S , with e_k as the eigenvector. Reversing this order of this chain shows the definition is unaffected by using J^t in place of J . We remark that k is the maximal length of a chain of L with eigenvector e_k , and also that K may have another

chain with eigenvalue λ . For example,

$$\begin{matrix} e_1 \\ e_2 \\ e_3 \end{matrix} \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

has $\langle e_1, e_2 \rangle$ and $\langle e_3 \rangle$ both as Jordan subspaces. Lemma 4.6, which states that a set of m chains with linearly independent eigenvectors is also linearly independent, will be used even before it is proven in Section 4.1.

In light of the above discussion, our goal is now to determine necessary and sufficient conditions for the existence of a full rank solution for ϕ a self-map of the ellipsoid $D\mathbb{B}^n$ fixing 0, $\phi'(0)$ an invertible, upper-triangular, Jordan matrix, and C_ϕ compact on $A_H^2(\mathcal{D})$. We now have a Jordan decomposition induced by $\phi'(0)^t$,

$$\mathbb{C}^n = S_1 \oplus \dots \oplus S_m,$$

and $\phi'(0)^t = \text{diag}(J_1, \dots, J_m)$ with each J_j a lower-triangular Jordan block. That is, the compression of $\phi'(0)^t$ to S_j is J_j , and S_j is a Jordan subspace of $\phi'(0)^t$. Let $n_j := \dim(S_j)$, so J_j is $n_j \times n_j$ with eigenvalue λ_j . It follows that $n_1 + \dots + n_m = n$. Notice also that the λ_j need not be distinct.

We will let the chain $e_1^j, \dots, e_{n_j}^j$ be our basis of S_j , so that

$$\{e_1^1, \dots, e_{n_1}^1, \dots, e_1^m, \dots, e_{n_m}^m\}$$

is our full basis on \mathbb{C}^n . Quite explicitly, we now have

$$J_j = \begin{matrix} e_1^j \\ e_2^j \\ \vdots \\ e_{n_j}^j \end{matrix} \begin{bmatrix} \lambda_j & & & \\ & 1 & & \\ & & \lambda_j & \\ & & \ddots & \ddots \\ & & & 1 & \lambda_j \end{bmatrix}.$$

Just as we represent $\phi'(0)^t$ in block form, we also write $F = (F_1, \dots, F_m)^t$ in block form, with each F_j a column-vector with length n_j . So, each

$$F_j = \begin{bmatrix} f_1^j \\ \vdots \\ f_{n_j}^j \end{bmatrix}$$

is a function from $\mathcal{D} \rightarrow \mathbb{C}^{n_j}$ with each $f_i^j : \mathcal{D} \rightarrow \mathbb{C}$. In other words, F_1 is the first n_1 components of F , then F_2 is the next n_2 components of

F , and so forth. Schroeder's equation becomes

$$\begin{bmatrix} F_1 \circ \phi \\ \vdots \\ F_m \circ \phi \end{bmatrix} = F \circ \phi = \phi'(0)F = \begin{bmatrix} J_1^t & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & J_m^t \end{bmatrix} \begin{bmatrix} F_1 \\ \vdots \\ F_m \end{bmatrix} = \begin{bmatrix} J_1^t F_1 \\ \vdots \\ J_m^t F_m \end{bmatrix}.$$

Thus, a Schroeder solution exists exactly when we can solve $F_j \circ \phi = J_j^t F_j$ for each j and the function $F = (F_j)^t$ has full rank near 0.

Unpacking notation we write $F_j = (f_1^j, \dots, f_{n_j}^j)^t$, and see $F_j \circ \phi = J_j^t F_j$ becomes

$$(5) \quad \begin{bmatrix} f_1^j \circ \phi \\ \vdots \\ f_{n_j-1}^j \circ \phi \\ f_{n_j}^j \circ \phi \end{bmatrix} = \begin{bmatrix} \lambda_j & 1 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \lambda_j & 1 \\ 0 & \dots & 0 & \lambda_j \end{bmatrix} \begin{bmatrix} f_1^j \\ \vdots \\ f_{n_j-1}^j \\ f_{n_j}^j \end{bmatrix} = \begin{bmatrix} \lambda_j f_1^j + f_2^j \\ \vdots \\ \lambda_j f_{n_j-1}^j + f_{n_j}^j \\ \lambda_j f_{n_j}^j \end{bmatrix}.$$

In other words, $f_1^j, \dots, f_{n_j}^j$ is a chain of C_ϕ with eigenvalue λ_j .

Lemma 3.2. *The following statements are equivalent.*

- (1) *There exists an $F_j = (f_1^j, \dots, f_{n_j}^j)^t$ with components linearly independent in \mathcal{H} satisfying $F_j \circ \phi = J_j^t F_j$.*
- (2) *$[\ker(C_\phi - \lambda_j I)] \cap [(C_\phi - \lambda_j I)^{n_j-1}(A_H^2(\mathcal{D}))]$ is nontrivial.*
- (3) *C_ϕ has a chain of length n_j and eigenvalue λ_j .*

Proof. The work above shows that $F_j = (f_1^j, \dots, f_{n_j}^j)^t$ satisfies $F_j \circ \phi = J_j^t F_j$ if and only if $\{f_i^j\}_i$ forms a chain of C_ϕ with eigenvalue λ . Lemma 4.6 shows the f_i^j are linearly independent. Thus (1) and (3) are equivalent. The equivalence of (2) and (3) is the definition of a chain. \square

We have reduced the problem of finding an F satisfying $F \circ \phi = \phi'(0)F$ to that of finding chains of C_ϕ which correspond to those of $\phi'(0)$. We now are ready to produce solutions (Section 4), and then discriminate which have full rank near 0 (Sections 5 and 6).

4. SOLUTIONS WITHOUT CONSIDERATION OF RANK NEAR 0

Forgetting the behavior of F near 0, we focus on a much more modest goal of producing a solution, F , whose component functions are linearly independent in $A_H^2(\mathcal{D})$. Unlike a full rank solution, a linearly independent solution necessarily exists (Theorems 4.14 and 4.15), yet, the process by which we develop a linearly independent solution will provide the groundwork for understanding exactly when we are equipped with a full rank solution. Specifically, we will use compactness of C_ϕ to reduce the problem to one of linear algebra, and Subsection 4.1, a

section of linear algebra theorems, provides the lion's share of work for both Sections 4 and 6.

The main technique in [CM03], as well as the current note is to exhibit a Bergman space on which C_ϕ is compact, and then use the compactness to produce our solutions. The following lemma and corollaries roughly say that any compact operator when viewed as an infinite matrix acts like a sufficiently large (finite) upper left corner. Consequently, we may reduce the problem of existence of a solution F to an analogous problem for an operator on \mathbb{C}^N .

Theorem 4.1. *Let C be a lower-triangular compact operator on a separable Hilbert space, \mathcal{H} . Given any $\lambda \neq 0$ the following are true.*

- (1) *We can write $\mathcal{H} = H_1 \oplus H_2$ with $H_1 \cong \mathbb{C}^N$ and*

$$C = \begin{bmatrix} U & 0 \\ V & W \end{bmatrix}$$

so that $|\lambda| > \|W\|$.

- (2) *$\ker(C - \lambda I) \cong \ker(U - \lambda I_N)$, where U is $N \times N$.*

It follows easily from this theorem that $\sigma(C) \setminus \{0\} = \text{diag}(C) \setminus \{0\}$, where $\sigma(C)$ denotes the spectrum of C . As these results are well known, we omit the proof. The details may be found in [CM03, Lemma 9]. The following corollary states the ideas of Theorem 4.1 in a way that is most useful for our purposes with explanations in the paragraphs that follow.

Corollary 4.2. *Let C be a lower-triangular compact operator on a separable Hilbert space, \mathcal{H} . Given any $\lambda \neq 0$ the following are true.*

- (1) *We can write $\mathcal{H} = H_1 \oplus H_2$ with $H_1 \cong \mathbb{C}^N$ and*

$$C = \begin{bmatrix} U & 0 \\ V & W \end{bmatrix}$$

so that $\lambda \notin \sigma_p(W)$, the eigenvalues of W .

- (2) *$\ker(C - \lambda I) \cong \ker(U - \lambda I_N)$, where U is $N \times N$.*

- (3) *Any formal solution, h to $(C - \lambda I)h = 0$ is in \mathcal{H} .*

Part (1) of the corollary below is stated for usability. In practice, given the lower triangular matrix C_ϕ and a fixed λ , it is much easier to choose the upper left corner U so that it contains every occurrence of λ on the diagonal (recall that C_ϕ is compact so the diagonal entries, i.e. its eigenvalues, converge to 0) than it is to choose U so that $\|W\| < |\lambda|$. As in the results of [CM03], to prove our main theorems we will choose U to include every eigenvalue of $\phi'(0)$ that appears on the diagonal of C_ϕ .

The important consequence of (2) is that given any

$$h_1 = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in \ker(U - \lambda I_N),$$

there exists a unique completion of this vector, that is, a unique

$$h_2 = \begin{bmatrix} x_{N+1} \\ \vdots \end{bmatrix} \in H_2$$

so that

$$h_1 + h_2 = \begin{bmatrix} x_1 \\ \vdots \\ x_N \\ x_{N+1} \\ \vdots \end{bmatrix} \in \ker(C_\phi - \lambda I).$$

By a formal solution, we mean a column vector $h = (x_1, x_2, \dots)^t$ which satisfies the equation $Ch = \lambda h$. No concern is given to the convergence of $h = \sum x_j e_j$ in \mathcal{H} . To see the use of (3), consider writing the equation $Ch = \lambda h$ with C as a given lower triangular matrix and λ also fixed. It gives an (infinite) system of equation, the first equation depending only on x_1 , the second depending on x_1 and x_2 and so on. If x_j can be chosen inductively to satisfy our algebraic requirements, then (3) tell us that $\sum x_j e_j$ converges in \mathcal{H} . In particular, when \mathcal{H} is a Hilbert space of analytic functions on a domain \mathcal{D} we see the x_j are the coefficients of the Taylor series of an analytic function, and (3) says that this corresponding Taylor series actually converges on \mathcal{D} .

We reiterate that the collection of ideas in Theorem 4.1 and Corollary 4.2 are well know and certainly the use of compactness of C_ϕ to find analytic solution is evident in [CM03]. Perhaps it is the thorough investigation of these ideas that allowed proofs of the main results of this paper.

As a last remark, we note that compactness is essential here. For example, the right and left shift operators are lower and upper triangular, respectively, and have the closed unit ball as their spectrum. In particular, the left shift has the entire open disk as it's point spectrum, but only 0 appears on their diagonal.

Let P be the orthogonal projection of $\mathcal{H} := A_H^2(\mathcal{D})$ to the subspace spanned by those multi-indices of order less than or equal to K ; that is,

$$P\mathcal{H} = \langle z_1, z_2, \dots, z_n, z_1^2, z_1 z_2, \dots, z_n^2, \dots, z_1^K, z_1^{K-1} z_2, \dots, z_n^K \rangle.$$

Now we may write

$$(6) \quad C_\phi = \begin{bmatrix} U & 0 \\ V & W \end{bmatrix}$$

so that $U = PC_\phi P$ with K sufficiently large so that λ_j does not occur on the diagonal of W for every $j = 1, \dots, m$. Following the notation above, let $H_1 \oplus H_2 = \mathcal{H}$ and $H_1 = P\mathcal{H}$ of dimension N , so that U is $N \times N$. We also let Q denote the orthogonal projection from \mathcal{H} to the first n coordinates. That is,

$$Q\mathcal{H} = \langle z_1, \dots, z_n \rangle,$$

and we note that $Qf = \nabla f(0)$ for any f in \mathcal{H} . Since $\phi'(0)^t$ is the upper left $n \times n$ corner of C_ϕ , we have $\phi'(0)^t = QC_\phi Q$, and

$$\langle z_1, \dots, z_n \rangle = S_1 \oplus \dots \oplus S_m,$$

our Jordan decomposition of $\phi'(0)^t$. Recall that the S_j are not necessarily orthogonal subspaces of \mathcal{H} , so we let Q_j be the projection to S_j along $(\oplus_{i \neq j} S_i) \oplus (I - Q)\mathcal{H}$. In particular, $Q = Q_1 + \dots + Q_m$.

Recall that a solution depends on the chains of C_ϕ . Now we use compactness to reduce the problem to understanding the chains of U , our sufficiently large upper left corner of C_ϕ . For notational ease, we will use “ e_j ” to denote chains of $\phi'(0)$ or $\phi'(0)^t$ (these are elements of \mathbb{C}^n), and we will use “ ϵ_j ” to denote chains of U (these are vectors in \mathbb{C}^N).

Lemma 4.3. *With the notation developed just above, suppose $0 \neq \lambda$ and $\lambda \notin \sigma(W)$. If $\epsilon_1, \dots, \epsilon_s$ is a chain of U with eigenvalue λ , then there is a unique chain g_1, \dots, g_s of C_ϕ with the same eigenvalue such that $Pg_j = \epsilon_j$. Conversely, if g_1, \dots, g_s is a chain of C_ϕ with eigenvalue λ , then $Pg_j = \epsilon_j$ is a chain of U with the same eigenvalue.*

Proof. Assume first that $\epsilon_1, \dots, \epsilon_s$ is a chain of U with eigenvalue λ ; that is

$$(7) \quad (U - \lambda I_N) : \epsilon_s \mapsto \dots \mapsto \epsilon_1 \mapsto 0.$$

An application of Corollary 4.2 furnishes the existence of a unique g_1 such that $Pg_1 = \epsilon_1$, and $(C - \lambda I)g_1 = 0$. Inductively, suppose that for each $1 \leq i \leq s - 1$ there are unique g_i such that $Pg_i = \epsilon_i$ and

$$(8) \quad (C_\phi - \lambda I)^k g_i = \begin{cases} 0 & \text{if } k \geq i \\ g_{i-k} & \text{if } k < i. \end{cases}$$

Now write $(C_\phi - \lambda I)^s = C + (-\lambda)^s I$ with $C = (C_\phi^s - s\lambda C_\phi^{s-1} + \dots + s(-\lambda)^{s-1} C_\phi)$. Since C_ϕ is compact and lower triangular, so also is C . An application of Corollary 4.2 to $C + (-\lambda)^s I$ produces a unique g_s

satisfying $Pg_s = \epsilon_s$ and $(C_\phi - \lambda I)^s g_s = (C + (-\lambda I)^s I)g_s = 0$. Next for any $1 \leq k \leq s-1$ we see $P(C_\phi - \lambda I)^k g_s = (U - \lambda I_N)^k \epsilon_s = \epsilon_{s-k}$. Uniqueness of g_{s-k} now implies that $g_{s-k} = (C_\phi - \lambda I)^k g_s$ completing our induction step.

For the converse suppose g_1, \dots, g_s form a chain of C_ϕ with eigenvalue λ . Clearly, $\epsilon_j := Pg_j$ satisfies (7). We must show $\epsilon_1 \neq 0$. But if $\epsilon_1 = 0$, then $\lambda g_1 = C_\phi g_1 = Wg_1$, a contradiction to $\lambda \notin \sigma(W)$. \square

Theorem 4.4. *With notation as above the following statements are equivalent.*

- (1) *There exists an $F_j = (f_1^j, \dots, f_{n_j}^j)^t$ with components linearly independent in \mathcal{H} satisfying $F_j \circ \phi = J_j^t F_j$.*
- (2) *$[\ker(U - \lambda_j I)] \cap [(U - \lambda_j I)^{n_j-1}(\mathcal{H})]$ is nontrivial.*
- (3) *U has a chain of length n_j and eigenvalue λ_j .*

Moreover, when these three conditions hold, $\epsilon_i^j := Pf_i^j$ is the chain in (3) and a chain ϵ_i^j of U uniquely determines such an $F_j = (f_i^j)^t$.

Proof. It follows from Lemma 3.2 that F_j is a solution if and only if the f_i^j form a chain. Suppose first $f_1^j, \dots, f_{n_j}^j$ is a chain of C_ϕ with eigenvalue λ_j , then clearly Pf_i^j , $i = 1, \dots, n_j$ is a chain of U . Conversely, given a chain of U , say $\epsilon_1^j, \dots, \epsilon_{n_j}^j$ with eigenvalue λ , Lemma 4.3 shows that there is a unique solution $F_j = (f_1^j, \dots, f_{n_j}^j)^t$ with $PF_i^j = \epsilon_i^j$. Furthermore, Lemma 4.6 shows that the f_i^j are linearly independent. This shows (1) \Leftrightarrow (3).

(2) \Leftrightarrow (3) is trivial. \square

The preceding results identify the existence of such an F_j with the chains of U . Notice that if U has a Jordan subspace of dimension at least n_j , then $[\ker(U - \lambda_j I)] \cap [(U - \lambda_j I)^{n_j-1}(\mathcal{H})]$ is indeed non-empty (and the desired F_j exists). We now turn our focus to the Jordan subspaces of U and state this in the next corollary.

Let ζ_1, \dots, ζ_l be the eigenvalues of U . Since U is similar to a Jordan form matrix, U has a Jordan decomposition,

$$\mathbb{C}^N = \tilde{S}_1 \oplus \dots \oplus \tilde{S}_l.$$

As usual, the Jordan subspace $\tilde{S}_k = \langle \{\epsilon_i^k\}_i \rangle$ with $1 \leq i \leq \dim(\tilde{S}_k)$ and ϵ_i^k a chain of U with eigenvalue ζ_k . Notice that each Jordan subspace of U contains exactly one eigenvector of U , and partially conversely, any eigenvector of U can lie in at most one \tilde{S}_k .

Corollary 4.5. *With notation as above, if there exists an \tilde{S}_k with $\zeta_k = \lambda_j$ (that is, the eigenvalue of S_k and the J_j coincide) and $\dim(S_k) \geq n_j$,*

then there is an $F_j = (f_i^j)^t$ satisfying $C_\phi F_j = J_j^t F_j$ with each linearly independent components.

In fact, the converse holds as well, giving necessary and sufficient conditions for such an F_j . See Theorem 4.13.

Proof. If there exists an S_k with dimension at least n_j and eigenvalue λ_j , we see it's eigenvector lives in $[\ker(U - \lambda_j)] \cap [(U - \lambda_j)^{n_j-1}(\mathcal{H})]$, which gives the result. \square

We are led to the following question:

Suppose we are given an $N \times N$ lower triangular matrix, U , with $\phi'(0)^t$ the upper left $n \times n$ corner. That is,

$$U = \begin{matrix} e_1 \\ \vdots \\ e_n \\ e_{n+1} \\ \vdots \\ e_N \end{matrix} \left[\begin{array}{ccc|ccc} J_1 & & & & & \\ & \ddots & & & & \\ & & J_m & & 0 & \\ \hline & & & Y & & Z \end{array} \right].$$

“Does each J_j appear, possibly longer, in the Jordan form of U ?” Equivalently, “Does there exist an injective map identifying each S_j (of a Jordan subspace decomposition for $\phi'(0)^t$) with an \tilde{S}_k (of a Jordan subspace decomposition for U) that shares the same eigenvalue as S_j and has possibly larger dimension?” Or in terms of chains, “For each j does there exist a chain, $\{e_1^j, \dots, e_{n_j}^j\}$, of U with eigenvalue λ_j , such that

$$\bigcup_{i,j} \{e_i^j\}$$

is a linearly independent set?” An affirmative answer will ensure the existence of a solution, F with linearly independent component functions. What follows is a subsection of linear algebra results that lead to this answer and are the basis of every major theorem of the paper.

4.1. Linear Algebra Results. Certainly some or all of the results in this section are known in one form or another. We develop and prove them here in a manner that is useful for our purposes.

Lemma 4.6. *Let L be a linear operator on a vector space V . Let $e_1^j, \dots, e_{n_j}^j$ be a chain of L with eigenvalue λ_j for each $1 \leq j \leq m$, and suppose the eigenvectors, $e_{n_1}^1, \dots, e_{n_m}^m$ are linearly independent. Then $\{e_i^j : 1 \leq j \leq m, 1 \leq i \leq n_j\}$ is a linearly independent set.*

Proof. We induct on $n = n_1 + \dots + n_m = |\{e_i^j : 1 \leq j \leq m, 1 \leq i \leq n_j\}|$. The result is clear for $n = 1$. Letting $n > 1$, we write

$$(9) \quad \sum_{j=1}^m \sum_{i=1}^{n_j} a_i^j e_i^j = 0$$

for constants a_i^j , and show that a_i^j are necessarily 0. Applying $L - \lambda_1 I$ gives

$$\sum_{i=1}^{n_1-1} a_i^j e_{i+1}^j + \sum_{j=2}^m \sum_{i=1}^{n_j} b_i^j e_i^j = 0$$

for some constants b_i^j . Since $e_2^1, \dots, e_{n_1}^1$ is still a chain of L , our inductive hypothesis implies $a_i^1 = 0$ for any $1 \leq i \leq n_1 - 1$. Repeating this argument, we have $a_i^j = 0$ for any $1 \leq i \leq n_j - 1$, and (9) becomes

$$\sum_{j=1}^m a_{n_j}^j e_{n_j}^j = 0.$$

The result follows from the linear independence of our eigenvalues, $e_{n_1}^1, \dots, e_{n_m}^m$. \square

Lemma 4.7. *Let J be a lower triangular λ -Jordan block, of length k . Set*

$$A = \begin{bmatrix} & & 0 \\ & J & \vdots \\ & & 0 \\ a_1 \dots a_k & a_{k+1} \end{bmatrix}.$$

If B is the Jordan-form matrix conjugate to A then either B has two Jordan blocks, namely J and the 1×1 matrix, $[a_{k+1}]$, or B is a λ -Jordan block of length $k + 1$. More precisely, the later case happens if and only if $a_{k+1} = \lambda$ and $a_k \neq 0$.

Proof. Let $\{e_1, \dots, e_{k+1}\}$ be the usual basis on \mathbb{C}^{k+1} . We will exhibit a new basis, $\{\epsilon_1, \dots, \epsilon_{k+1}\}$ giving the desired result.

Suppose first that $a_{k+1} \neq \lambda$. Set $\epsilon_{k+1} = e_{k+1}$, and $\epsilon_j = e_j + c_j e_{k+1}$ with $c_k = a_k / (\lambda - a_{k+1})$ and $c_j = (c_{j+1} - a_j) / (a_{k+1} - \lambda)$. It follows that ϵ_k and ϵ_{k+1} are eigenvectors with eigenvalues λ and a_{k+1} , respectively, and $A(\epsilon_j) = \lambda \epsilon_j + \epsilon_{j+1}$ for each $1 \leq j \leq (k-1)$. So $B = \text{diag}(J, a_{k+1})$.

Now let $a_{k+1} = \lambda$. Set $\epsilon_{k+1} = e_{k+1}$, $\epsilon_1 = e_1$, and $\epsilon_j = e_j + a_{j-1} e_{k+1}$ for each $2 \leq j \leq k$. It follows that $A(\epsilon_j) = \lambda \epsilon_j + \epsilon_{j+1}$ for each $1 \leq j \leq (k-1)$. If $a_k = 0$, we see ϵ_k and ϵ_{k+1} are linearly independent eigenvectors, and again we have $B = \text{diag}(J, a_{k+1}) = \text{diag}(J, \lambda)$. On the

other hand, if $a_k \neq 0$, $A(\epsilon_k) = \epsilon_{k+1}$ and ϵ_{k+1} is the lone eigenvector. It follows that B is a $(k+1) \times (k+1)$ λ -Jordan block. \square

Lemma 4.8. *Let*

$$A = \begin{array}{c} e_1 \\ \vdots \\ e_n \\ e_{n+1} \\ \vdots \\ e_{n+k} \\ e_{n+k+1} \end{array} \left[\begin{array}{ccc|ccc|c} & & & & & & 0 \\ & & & & & & \vdots \\ & & & & & & 0 \\ \hline & & & & & & 0 \\ & & & & & & \vdots \\ & & & & & & 0 \\ \hline 0 & \dots & 0 & 1 & 0 & \dots & 0 & 1 & \lambda \end{array} \right]$$

so that the last row of A has a 1 in columns n and $n+k$, λ in column $n+k+1$, and all other entries 0. J_1 and J_2 are both lower triangular λ -Jordan blocks of length n and k , respectively with $k \geq n$. Then the Jordan-form matrix conjugate to A consists of two λ -Jordan blocks of length n and $k+1$ respectively.

Proof. Set $\epsilon_1 = e_1 - e_{k+1}$, $\epsilon_2 = e_2 - e_{k+2}$, ..., $\epsilon_n = e_n - e_{n+k}$, and let $\epsilon_{n+j} = e_{n+j}$ for $1 \leq j \leq (k+1)$. Notice that $\{\epsilon_1, \dots, \epsilon_{n+k+1}\}$ is linearly independent. Expressing A in this basis gives the result with $\langle \epsilon_1, \dots, \epsilon_n \rangle$ as the Jordan subspace with dimension n , and $\langle \epsilon_{n+1}, \dots, \epsilon_{n+k+1} \rangle$ the Jordan subspace with dimension $k+1$. \square

Theorem 4.9. *Let $C = \text{diag}(J_1, \dots, J_m)$ be an $n \times n$ matrix with each J_j a lower triangular Jordan block with eigenvalue λ_j . Let*

$$A = \begin{bmatrix} C & 0 \\ D & E \end{bmatrix}$$

an $N \times N$ lower triangular matrix and let B be a Jordan form matrix similar to A . Then for every Jordan block, J_j , of C , there is a corresponding Jordan block, \tilde{J}_j of B sharing the eigenvalue λ_j , and such that $\text{length}(J_j) \leq \text{length}(\tilde{J}_j)$.

Proof. Inductively, it suffices to show the result for the case D is an $N \times 1$ row vector and $E = (\lambda)$ is 1×1 , i.e.

$$A = \left[\begin{array}{ccc|c} & & & 0 \\ & & & \vdots \\ & & & 0 \\ \hline C & & & \\ \hline a_1 & \dots & a_N & \lambda \end{array} \right].$$

As above let J_j be $n_j \times n_j$ with eigenvalue λ_j , and $\{e_1, \dots, e_N\}$ the usual basis on \mathbb{C}^N .

Focus first on J_1 . In the case $\lambda_1 \neq \lambda$ or the case $\lambda_1 = \lambda$ and $a_{n_1} = 0$, the proof of Lemma 4.7 allows us to assume $a_1 = \dots = a_{n_1} = 0$ by replacing e_j with $\epsilon_j = e_j + c_j e_{N+1}$ for $1 \leq j \leq n_1$ with appropriate values of c_j . If, on the other hand, we have $\lambda_1 = \lambda$ and $a_{n_1} \neq 0$, a similar application of Lemma 4.7 gives the simplification $a_1 = \dots = a_{n_1-1} = 0$, and $a_{n_1} = 1$ with no loss of generality.

Consequently, it suffices to show the result under the assumption that each J_j is a λ -Jordan block, and furthermore that

$$a_i = \begin{cases} 0 & \text{if the } i^{\text{th}} \text{ column is not the last column of some } J_j \\ 1 & \text{if the } i^{\text{th}} \text{ column is the last column of some } J_j. \end{cases}$$

Explicitly,

$$A = \left[\begin{array}{ccc|ccc|ccc|c} & & & & & & & & & 0 \\ & & & & & & & & & \vdots \\ & & & & & & & & & 0 \\ & & & & & & & & & \vdots \\ & & & & & & & & & 0 \\ & & & & & & & & & \vdots \\ & & & & & & & & & 0 \\ & & & & & & & & & \vdots \\ & & & & & & & & & 0 \\ 0 & \dots & 0 & 1 & \dots & 0 & \dots & 0 & 1 & \lambda \end{array} \right]$$

By using a unitary change of basis, arrange the Jordan blocks so the $n_i = \text{length}(J_i) \leq n_j = \text{length}(J_j)$ for $1 \leq i \leq j \leq m$. Inductive application of 4.8 will give the appropriate change of basis to attain the Jordan form matrix, B , and moreover, each J_i will appear unaffected in B except for J_m which will now appear longer. \square

We are now able to answer our question. When $A = U$ and $C = \phi'(0)^t = \text{diag}(J_1, \dots, J_m)$, the above result says that there is a basis so that expressing U in this basis gives the Jordan matrix, $U = \text{diag}(\tilde{J}_1, \dots, \tilde{J}_l)$ with $l \geq m$. Moreover, for every $1 \leq j \leq m$, $\text{length}(J_j) \leq \text{length}(\tilde{J}_j)$ and the eigenvalue of \tilde{J}_j is λ_j .

Corollary 4.10. *Let*

$$\mathbb{C}^n = \bigoplus_{j=1}^m S_j \text{ and } \mathbb{C}^N = \bigoplus_{j=1}^l \tilde{S}_j$$

be Jordan subspace decompositions of $\phi'(0)^t$ and U , respectively. Then there is an injection,

$$\tau : \{S_j\}_{j=1}^m \rightarrow \{\tilde{S}_j\}_{j=1}^l,$$

such that the eigenvalue of S_j is preserved, and $\dim(S_j) \leq \dim(\tau S_j)$.

The following theorem pays close attention to the proof of Theorem 4.9 and will be useful in distinguishing whether a solution with full rank near 0 exists or not. The notation introduced in the theorem below matches that developed earlier in the section.

Theorem 4.11. *Let $\mathbb{C}^n = \bigoplus_{j=1}^m S_j$ be a Jordan subspace decompositions induced by $\phi'(0)^t$, and $S_j = \langle e_i^j : 1 \leq i \leq n_j \rangle$ the usual chain basis. Consider \mathbb{C}^n as the subspace of \mathbb{C}^N spanned by the first n components, and let Q_j be the projection to S_j along $(\bigoplus_{i \neq j} S_i) \oplus (\mathbb{C}^n)^\perp$. Then there exists a Jordan subspace decomposition induced by U , namely, $\mathbb{C}^N = \bigoplus_{j=1}^l \tilde{S}_j$ with $d_j = \dim(\tilde{S}_j)$ and chain bases $\langle e_i^j : 1 \leq i \leq d_j \rangle = \tilde{S}_j$ such that the following hold.*

- (1) $Q_j(e_i^j) = e_i^j$ for any $1 \leq j \leq m$, and $1 \leq i \leq n_j$.
- (2) The map $\tau(S_j) = \tilde{S}_j$ satisfies the result of Corollary 4.10.
- (3) $d_j > n_j$ for some $1 \leq j \leq m$ if and only if $Q(\ker(U - \lambda_j I_N)) \not\subseteq \ker(\phi'(0)^t - \lambda_j I_n)$.

Proof. The basis $\{\epsilon_i^j\}$ of \mathbb{C}^N which will give our solution is the basis that is produced in the proof of Theorem 4.9. To prove this, we will prove our results hold at each step of the induction proof of Theorem 4.9. With no loss of generality, we will show it only for the first step. Assume we have

$$U_{n+1} = \left[\begin{array}{ccc|c} & & & 0 \\ & \phi'(0)^t & & \vdots \\ & & & 0 \\ \hline d_1 & \dots & d_n & \lambda \end{array} \right],$$

acting on $\mathbb{C}^n \oplus \mathbb{C}$, and denote our basis elements for \mathbb{C}^n $\{e_i^j : 1 \leq j \leq m, 1 \leq i \leq n_j\}$ (in chains as above) and e_{n+1} the extra basis vector.

(1) In order to put U_{n+1} into Jordan form, the basis elements are replaced via applications of Lemmas 4.7 and 4.8. In both of these Lemmas' proofs we see a basis vector e_i^j is always replaced with $e_i^j + x$ where x is some vector satisfying $Q_j x = 0$. This proves (1).

(2) Notice that a chain of basis elements, say $e_1^j, \dots, e_{n_j}^j$ always is replaced by a new chain with the same eigenvalue and of possibly greater length. This proves (2).

(3) When putting U_{n+1} into Jordan form, at most one chain can gain length. Suppose first that some chain, say $e_1^j, \dots, e_{n_j}^j$, gains length. So j is fixed for the moment. It follows from the proofs of Lemmas 4.7 and 4.8 that the new chain has eigenvector e_{n+1} , and any other chain's original eigenvector, say $e_{n_k}^k$, is replaced with $e_{n_k}^k + c_k e_{n+1} + b_k e_{n_j}^j$ where b_k is either 1 or 0, and c_k some constant. It follows

that $\dim(Q[\ker(U_{n+1} - \lambda_j I_{n+1})]) = \dim[\ker(\phi'(0)^t - \lambda_j I_n)] - 1$ and so $Q[\ker(U_{n+1} - \lambda_j I_{n+1})] \subsetneq \ker(\phi'(0)^t - \lambda_j I_n)$.

Lastly, suppose now that no chain gains length. Lemma 4.8 is not used, and from 4.7 it follows that each original eigenvector, $e_{n_j}^j$ is replaced with $e_{n_j}^j + c_{j,n_j} e_{n+1}$. Consequently, $(Q \ker(U_{n+1} - \lambda_j I_{n+1})) = \ker(\phi'(0)^t - \lambda_j I_n)$. \square

The following generalization of 4.9 is now easily within our grasp.

Theorem 4.12. *Let*

$$A = \begin{bmatrix} C & 0 \\ D & E \end{bmatrix} \text{ or } \begin{bmatrix} C & D \\ 0 & E \end{bmatrix},$$

an $N \times N$ matrix, with C the upper left $n \times n$ corner of A and $n \leq N$. Let

$$\mathbb{C}^n = \oplus_{j=1}^m S_j \text{ and } \mathbb{C}^N = \oplus_{j=1}^l \tilde{S}_j$$

be Jordan subspace decompositions of C and A , respectively. Then there is an injection,

$$\tau : \{S_j\}_{j=1}^m \rightarrow \{\tilde{S}_j\}_{j=1}^l,$$

such that the eigenvalue of S_j is preserved, and $\dim(S_j) \leq \dim(\tau S_j)$.

Proof. We will show the result for

$$A = \begin{bmatrix} C & 0 \\ D & E \end{bmatrix},$$

as a proof for the case

$$\begin{bmatrix} C & D \\ 0 & E \end{bmatrix}$$

is similar. Let M be an invertible matrix such that MCM^{-1} is in Jordan form (lower triangular), and let L be an invertible matrix such that LEL^{-1} is lower triangular. Set

$$\tilde{M} = \begin{bmatrix} M & 0 \\ 0 & L \end{bmatrix},$$

so

$$\tilde{M}A\tilde{M}^{-1} = \begin{bmatrix} MCM^{-1} & 0 \\ \tilde{D} & LEL^{-1} \end{bmatrix}$$

is a lower triangular matrix with upper left $n \times n$ corner in Jordan form. It follows that $\{MS_j\}_{j=1}^m$ and $\{\tilde{M}\tilde{S}_j\}_{j=1}^l$ are the Jordan spaces of MCM^{-1} and $\tilde{M}A\tilde{M}^{-1}$, respectively, with unchanged eigenvalues and dimensions. An application of Corollary 4.10 gives the result. \square

4.2. Main Results of Section 4. We can now state and prove the main theorems of the section but first we give the promised converse to Corollary 4.5. We will continue to use the notation developed throughout the section.

Theorem 4.13. *There is an $F_j = (f_1^j, \dots, f_{n_j}^j)^t$ with linearly independent components satisfying $C_\phi F_j = J_j^t F_j$ if and only if U has a Jordan space decomposition, $\mathbb{C}^N = \bigoplus_{j=1}^l \tilde{S}_j$ with some S_k of dimension greater than or equal to n_j , and with eigenvalue λ_j .*

Proof. As the converse is given by Corollary 4.5 we only provide the forward direction.

Suppose F_j is a solution with linearly independent component functions. It follows that $\{\epsilon_i = P f_i^j : 1 \leq i \leq n_j\}$ is a chain of U with length n_j and eigenvalue λ_j . By Lemma 4.6 we know the ϵ_i are linearly independent, so we may extend this chain to a basis of \mathbb{C}^N . Expressing U in this basis we see U is of the form

$$\begin{bmatrix} J & A \\ 0 & B \end{bmatrix},$$

with the Jordan block J the compression of U to $\langle \epsilon_1, \dots, \epsilon_{n_j} \rangle$. An application of Theorem 4.12 gives the result. \square

While we now have necessary and sufficient conditions for a solution with linearly independent components, our jubilation is postponed as the achievement is overshadowed by the main results of the section.

Theorem 4.14. *Let ϕ be an analytic map on the ellipsoid \mathcal{D} fixing 0 with $\phi'(0)$ a full rank upper-triangular Jordan form matrix. Suppose also that C_ϕ is compact on a Bergman space, $A_H^2(\mathcal{D})$. Then there is an analytic F satisfying $F \circ \phi = \phi'(0)F$ such that the component functions of F are linearly independent in $A_H^2(\mathcal{D})$.*

Proof. As before, $\phi'(0)^t = \text{diag}(J_1, \dots, J_m)$, and $A_H^2(\mathcal{D}) = \mathcal{H} = H_1 \oplus H_2$ with

$$H_1 = \langle z_1, z_2, \dots, z_n, z_1^2, z_1 z_2, \dots, z_n^2, \dots, z_1^K, z_1^{K-1} z_2, \dots, z_n^K \rangle.$$

The projection to H_1 is denoted P , and H_1 is isomorphic to \mathbb{C}^N . Further, we write

$$C_\phi = \begin{bmatrix} U & 0 \\ V & W \end{bmatrix},$$

with $U = PC_\phi P$ and K sufficiently large so that λ_j does not appear on the diagonal of W for every $j = 1, \dots, m$.

Let $\{S_j\}_{j=1}^m$ and $\{\tilde{S}_j\}_{j=1}^l$ be as in Theorem 4.11, whose respective bases are the chains, $e_1^j, \dots, e_{n_j}^j$ and $\epsilon_1^j, \dots, \epsilon_{d_j}^j$. Recall that $d_j = \dim(\tilde{S}_j) \geq \dim(S_j) = n_j$ for each $1 \leq j \leq m$, and that $\{\epsilon_i^j : 1 \leq i \leq d_j, 1 \leq j \leq l\}$ is a basis for H_1 .

Lemma 4.3 furnishes us with unique $F_j = (f_1^j, \dots, f_{n_j}^j)^t$ such that $C_\phi F_j = J_j^t F_j$ and $P f_i^j = \epsilon_{d_j - n_j + i}^j$. Since $\{\epsilon_i^j : 1 \leq i \leq d_j, 1 \leq j \leq l\}$ is linearly independent, we see $\{f_i^j : 1 \leq i \leq n_j, 1 \leq j \leq l\}$ is, too. \square

Using Section 2 we can reformulate Theorem 4.14 in terms of our original hypotheses for Schroeder's equation on the ball \mathbb{B}^n and are rewarded with the following result.

Theorem 4.15 (Main Theorem of Section 4). *Let ϕ be a self-map of \mathbb{B}^n fixing 0 with $\phi'(0)$ invertible and ϕ not unitary on any slice. Then there exists an analytic F satisfying $C_\phi F = \phi'(0)F$ and having component functions that are linearly independent in $A_G^2(\mathbb{B}^n)$.*

Unfortunately, the fact that F has linearly independent component functions does not imply that $F(\{|z| < \epsilon\})$ is open in \mathbb{C}^n for small ϵ (as is the case when $F'(0)$ is full rank). For example, $(z_1, z_2) \mapsto (z_1, z_1^2)$ contains no open subset of \mathbb{C}^2 in its image, while $(z_1, z_2) \mapsto (z_1, z_2^2)$ does. Additionally, neither of these maps have full rank near 0, but both do have linearly independent component functions.

Example 4.16.

Consider $\phi(z_1, z_2) = (z_1/2, z_2/4 + z_1^2/16)$ on \mathbb{B}^2 . Clearly $\phi(0) = 0$ and $|\phi(z)|^2 \leq |z_1|^2/4 + 4(|z_2|^2/16 + |z_1|^4/16^2) < |z|$ for $0 < |z| < 1$. Now,

$$\phi'(0)^t = \text{diag}(1/2, 1/4),$$

and

$$C_\phi = \begin{bmatrix} 1/2 & 0 & 0 & 0 & \dots \\ 0 & 1/4 & 0 & 0 & \dots \\ 0 & 1/16 & 1/4 & 0 & \dots \\ 0 & 0 & 0 & 1/8 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

so $\ker(C_\phi - (I/4)) = \langle z_1^2 \rangle$. $\phi'(0)$ has two chains both of length 1, and eigenvalues $1/2$ and $1/4$, respectively. C_ϕ has a $(1/2)$ -chain of length 1, and a $(1/4)$ -chain of length 2, namely $16z_2 \mapsto z_1^2 \mapsto 0$. Thus, $(az_1, bz_1^2)^t$ with a and b fixed complex numbers are the only solutions to Schroeder's equation. Consequently, every such solution must map open subsets of \mathbb{B}^2 into a 1-dimensional manifold. This also implies

that no full rank solution can exist, but certainly $(z_1, z_1^2)^t$ is a solution with linearly independent components.

We invite the interested reader to consult [CM03, §4], a section dedicated to similar examples.

This section showed that the existence of a solution, F , with linearly independent components was characterized by the Jordan subspaces of U in relation to those $\phi'(0)^t$, and hence, that such a solution always exists (Theorem 4.14). While the harder problem of finding a full rank solution is expectedly less generous, in retrospect we know how to go about it. First we find the appropriate chains of U (or equivalently C_ϕ) and then we check to see if their projections to $\langle z_1, \dots, z_n \rangle$ form a linearly independent set. Before delving into this topic, we investigate known obstructions to a full rank solution, and the role of resonant eigenvalues.

5. RESONANCE

Following the terminology of the literature, we say λ_j is a “resonant eigenvalue” of $\phi'(0)$ if (and only if) $\lambda_j = \lambda_1^{k_1} \dots \lambda_m^{k_m}$ for some non-negative integers k_i satisfying $\sum k_i > 1$ (for example [Eno07, p.2]). If we assume $\phi'(0)$ is upper triangular (or better in Jordan form) so that C_ϕ is lower triangular, then we notice that λ_j appears on the diagonal of C_ϕ below $\phi'(0)^t$ if and only if it is a resonant eigenvalue. In [CM03], Cowen and MacCluer identify that such resonance can cause an obstruction to finding a full rank solution. On the other hand, examples of a full-rank solution in the presence of resonance are given [CM03, Examples 2,3].

Subsequently, [Eno07] addresses Schroeder’s equation in several variables from a completely algebraic point of view, representing all functions as formal power series. In other words, the coordinate functions of ϕ and F are considered as column vectors (with the same notation as the current paper) but with no consideration of convergence of the corresponding Taylor series.

Theorem 5.1. [Eno07, Theorem 4.3] *Let ϕ be an analytic self-map of the ball, fixing 0, $\phi'(0)$ non-singular, and ϕ not unitary on any slice. If ϕ has no resonant eigenvalues, then there exists a formal power series solution to Schroeder’s equation with full rank near 0.*

To understand this fully, write $F = (f_1, \dots, f_n)^t$, and then write

$$f_j = \begin{bmatrix} x_1^j \\ x_2^j \\ \vdots \end{bmatrix}.$$

Now, writing $F \circ \phi = \phi'(0)F$ and $F'(0) = I_n$ gives an infinite system of equations with the x_i^j as the unknowns. In the absence of resonance, Enoch has shown that the x_i^j can be found to satisfy these equations. The issue which remains is of course whether the corresponding power series for each f_j actually converges on \mathbb{B}^n .

Theorem 5.2. *Let ϕ be an analytic self-map of the ball, fixing 0, $\phi'(0)$ non-singular, and ϕ not unitary on any slice. Any formal power series F satisfying $C_\phi F = \phi'(0)F$ is indeed analytic on \mathbb{B}^n .*

This theorem follows directly from the uniqueness statement of Theorem 4.4. In particular, Corollary 4.2 and the following paragraphs facilitate this line of reasoning. When $\phi'(0)$ is upper triangular and C_ϕ lower triangular (which can always be assumed with no loss of generality), the infinite system of equations written at once by Schroeder's equation can be solved inductively. In fact, Theorem 4.14 assures us that at least one solution exists, although there may be many as in [CM03, §Example 2], and Theorem 5.2 assures us all solutions found in this purely algebraic manner are indeed analytic. In light of this result, Enoch's algebraic method may be the best way to find a solution in practice.

Corollary 5.3. *Let ϕ be an analytic self-map of the ball, fixing 0, $\phi'(0)$ non-singular, and ϕ not unitary on any slice. If ϕ has no resonant eigenvalues, then a full rank solution to Schroeder's equation exists.*

We have obtained that resonance is necessary whenever there is no full rank solution, but what is obscure is why resonance is not sufficient to prevent a solution (recall that [CM03] provides examples of a full rank solution in the presence of resonance.) To illustrate this phenomenon, we provide an example similar to 4.16 which has a full rank solution.

Example 5.4. Now set $\phi(z_1, z_2) = (1/2)(z_1, z_2/2)$. Similar to the 4.16, it is easy to see that ϕ satisfies our hypotheses with domain \mathbb{B}^2 and experiences the same resonance. We now have

$$C_\phi = \begin{array}{c} z_1 \\ z_2 \\ z_1^2 \\ z_1 z_2 \\ \vdots \end{array} \begin{array}{ccccc} \phi_1 & \phi_2 & \phi_1^2 & \phi_1 \phi_2 & \dots \\ \left[\begin{array}{ccccc} 1/2 & 0 & 0 & 0 & \dots \\ 0 & 1/4 & 0 & 0 & \dots \\ 0 & 0 & 1/4 & 0 & \dots \\ 0 & 0 & 0 & 1/8 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right] \end{array} = \text{diag}\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \dots\right)$$

so it is easy to see that $f_1 = z_1$, and $f_2 = z_2$ are eigenfunctions with eigenvalues $1/2$ and $1/4$, respectively. Thus $F = (af_1, bf_2)^t$ with any nonzero a and b gives a full rank solution.

In both examples $\phi'(0)^t$ has one $(1/4)$ -Jordan block of length 1, but C_ϕ has one $(1/4)$ -Jordan block of length 2 in Example 4.16 and two $(1/4)$ -Jordan blocks of length 1 in Example 5.4. It is the fact that the original $(1/4)$ -Jordan block of $\phi'(0)^t$ gained length in the first example that forces any $(1/4)$ -eigenfunction to have both first derivatives 0, thereby preventing a full rank solution. Resonance is only responsible for the appearance of $1/4$ on the diagonal of C_ϕ below $\phi'(0)^t$.

We shall see that the only possible obstruction to a full rank solution is illustrated by the Example 4.16. Roughly speaking, what goes wrong is that if an original Jordan block of $\phi'(0)^t$ gains length when U is put into Jordan form, then it is necessary for the corresponding eigenfunction to have all of its first derivatives 0. Since a solution F must include an eigenfunction for each original Jordan block of $\phi'(0)^t$, any such solution F cannot be full rank.

6. CRITERIA FOR A FULL RANK SOLUTION

By contrast to this paper, [CM03] assumes with no loss of generality that $\phi'(0)$ is upper triangular by conjugating ϕ with a unitary matrix. Since a unitary change of basis preserves the ball and even the norm in $A_H^2(\mathcal{D})$, few of the concerns of Section 2 arise. The main theorem of [CM03] is that if $\phi'(0)$ is diagonalizable (in addition to our hypotheses), then a full rank Schroeder solution exists if and only if $\dim[\ker(C_\phi - \mu I)]$ is the number of times μ appears on the diagonal of C_ϕ . Equivalently, that U is also diagonalizable precisely when a full rank solution exists. Our main theorem, (Theorem 6.3) is an extension of this result to the general case.

While in Section 4 it was useful to decompose \mathbb{C}^n into the Jordan subspaces, in the current section it will be useful to consider generalized eigenspaces. Recall that if L is a linear operator on a vector space X , for each distinct eigenvalue, μ , the corresponding generalized eigenspace is

$$T_\mu = \bigcup_{k=1}^{\dim(X)} \ker[(L - \mu I)^k].$$

In our case, $L = \phi'(0)^t$ with indistinct eigenvalues $\lambda_1, \dots, \lambda_m$. Thus we let μ_1, \dots, μ_k be the distinct eigenvalues of $\phi'(0)^t$, and denote the generalized eigenspace decomposition $\mathbb{C}^n = T_{\mu_1} \oplus \dots \oplus T_{\mu_k}$.

We will denote the compression of $\phi'(0)^t$ to T_{μ_j} by L_j , so that in block form $\phi'(0)^t = \text{diag}(L_1, \dots, L_k)$, and each L_j is just the Jordan

form matrix with those Jordan blocks of $\phi'(0)^t$ that have μ_j on their diagonal. Correspondingly, we write $F = (F_{\mu_j})^t$, in block form. For example, if we assume $\lambda_1, \dots, \lambda_s$ are the only eigenvalues equal to μ_1 , we have $L_1 = \text{diag}(J_1, \dots, J_s)$, and $F_{\mu_1} = (F_1, \dots, F_s)^t$ with F_j as defined in Section 4. Now, $F \circ \phi = \phi'(0)F$ becomes the system of equations

$$(10) \quad F_{\mu_j} \circ \phi = L_j^t F_{\mu_j}$$

for $1 \leq j \leq k$.

The advantage of reorganizing into generalized eigenspaces is that we can reduce the problem to finding a full rank solution F_{μ_j} for each j . Specifically, if each F_{μ_j} can be found to satisfy (10) and have full rank near 0, then Proposition 6.1 shows that F is indeed a full rank solution.

Proposition 6.1. *If Equation (10) is satisfied and each F_{μ_j} has linearly independent component functions, then F has linearly independent component functions. In particular, F is a full rank solution if and only if Equation (10) holds with each F_{μ_j} full rank near 0.*

Proof. Since each coordinate function of F_{μ_j} is a member of the generalized eigenspace of C_ϕ with eigenvalue μ_j and the μ_j are distinct, the component functions of F are linearly independent as soon as the component functions of each F_{μ_j} are. \square

Thus we fix an eigenvector μ and seek a full rank solution to $F_\mu \circ \phi = L_\mu^t F_\mu$.

Theorem 6.2. *Let ϕ be an analytic map on the ellipsoid \mathcal{D} fixing 0 with $\phi'(0)$ a full rank upper-triangular Jordan form matrix. Suppose also that C_ϕ is compact on the Bergman space, $\mathcal{H} = A_H^2(\mathcal{D})$ as defined above. A full rank solution F_μ exists if and only if $Q(\ker(U - \mu I_N)) = \ker(\phi'(0)^t - \mu I_n)$, with Q the orthogonal projection to $\langle z_1, \dots, z_n \rangle$. Consequently, a full rank solution F exists if and only if $Q(\ker(U - \mu_j I_N)) = \ker(\phi'(0)^t - \mu_j I_n)$ for every eigenvalue μ_j of $\phi'(0)^t$.*

Proof. With no loss of generality, let $\mu = \lambda_1 = \dots = \lambda_s$ and assume $\lambda_j \neq \mu$ for any $s < j \leq m$. Thus, the generalized eigenspace of $\phi'(0)^t$ with eigenvalue μ is $T := T_\mu = S_1 \oplus \dots \oplus S_s$, and as before we have the chain $e_1^j, \dots, e_{n_j}^j$ forming a basis of S_j with $n_j = \dim(S_j)$. Recall that since Q is the projection from \mathcal{H} to $\langle z_1, \dots, z_n \rangle$, $Qg = \nabla g(0)$ for any g in \mathcal{H} . So,

$$F_\mu = [F_1, \dots, F_{m_0}]^t = [(f_1^1, \dots, f_{n_1}^1), (f_1^2, \dots, f_{n_2}^2), \dots, (f_1^s, \dots, f_{n_s}^s)]^t$$

has full rank near 0 if and only if $\{Qf_i^j : 1 \leq j \leq s, 1 \leq i \leq n_j\}$ is linearly independent.

Suppose $Q(\ker(U - \mu I_N)) = \ker(\phi'(0)^t - \mu I_n)$. There exist Jordan decompositions $\mathbb{C}^n = \oplus_{j=1}^n S_j$ and $\mathbb{C}^N = \oplus_{j=1}^l \tilde{S}_j$ for $\phi'(0)^t$ and U respectively, satisfying the result of Theorem 4.11. That is, for each $1 \leq j \leq s$ the chains $e_1^j, \dots, e_{n_j}^j$ and $\epsilon_1^j, \dots, \epsilon_{n_j}^j$ are bases for S_j and \tilde{S}_j respectively, and they satisfy $Q_j \epsilon_i^j = e_i^j$. Now Theorem 4.4 gives unique $F_j = (f_1^j, \dots, f_{n_j}^j)^t$ satisfying both $F_j \circ \phi = J_j^t F_j$ and $P(f_i^j) = \epsilon_i^j$. Since $Q_j(f_i^j) = Q_j(\epsilon_i^j) = e_i^j$, we have $Q(f_i^j) \neq 0$. It follows that $Q(f_1^j), \dots, Q(f_{n_j}^j)$ is a chain of $\phi'(0)^t$ for each j . Lemma 4.6 shows that $\{Qf_i^j : 1 \leq j \leq s, 1 \leq i \leq n_j\}$ is a linearly independent set, which is equivalent to F_μ having full rank.

Conversely, suppose that F_μ is a full rank solution. This implies that $\ker(\phi'(0)^t - \mu I_n) = \langle Qf_{n_j}^j : 1 \leq j \leq s \rangle$. Since $Pf_{n_j}^j$ with $1 \leq j \leq s$ are eigenvectors of U , we have the result. \square

Section 2 now allows us to formulate the result under the general hypotheses, that is, for ϕ an analytic self-map of \mathbb{B}^n , fixing 0, not unitary on any slice, and with full rank near 0. The theorem above applies to $D\phi D^{-1}$, and

$$C_{D\phi D^{-1}} = \begin{bmatrix} U & 0 \\ V & W \end{bmatrix},$$

where D is chosen so that $(D\phi D^{-1})'(0)$ is an upper triangular Jordan form matrix. So, to state the theorem in terms of ϕ we just need to translate “ $Q[(\ker(U - \mu I_N))] = \ker((D\phi'(0)^t D^{-1})^t - \mu I_n)$ ” to the language of C_ϕ . Recall that C_ϕ is compact on $A_G^2(\mathbb{B}^n)$, and $C_{D\phi D^{-1}}$ is compact on $A_H^2(D\mathbb{B}^n)$, where $\iota(f) = f \circ D^{-1}$ is an isometric isomorphism from $A_G^2(\mathbb{B}^n)$ to $A_H^2(D\mathbb{B}^n)$. Write

$$C_\phi = \begin{bmatrix} \tilde{U} & 0 \\ \tilde{V} & \tilde{W} \end{bmatrix},$$

in block form acting on $A_G^2(\mathbb{B}^n) = \tilde{H}_1 \oplus \tilde{H}_2$, with $\tilde{H}_1 = \langle z^\alpha \in A_G^2(\mathbb{B}^n) : |\alpha| \leq K \rangle$ and K large enough that \tilde{W} shares no eigenvalue with $\phi'(0)^t$. (It should be noted that the general C_ϕ is block lower triangular, as evidenced above. To convince oneself of this, notice that ϕ^α cannot have a nonzero derivative of order less than $|\alpha|$.) Since ι preserves homogeneous polynomials, we see that $\iota(\tilde{H}_1) = H_1$, and $\iota(\tilde{H}_2) = H_2$. Consequently, $\tilde{U} = \iota^{-1}U\iota$, and $Q(\ker(U - \mu I_N)) = \ker((D\phi'(0)^t D^{-1})^t - \mu I_n)$ is equivalent to $\tilde{Q}(\ker(\tilde{U} - \mu I_N)) = \ker(\phi'(0)^t - \mu I_n)$, where \tilde{Q} is the orthogonal projection from $A_G^2(\mathbb{B}^n)$ to $\langle z_1, \dots, z_n \rangle$. Thus a full rank solution, F_μ exists if and only if $\tilde{Q}(\ker(\tilde{U} - \mu I_N)) = \ker(\phi'(0)^t - \mu I_n)$.

Since in general,

$$U = \begin{bmatrix} \phi'(0)^t & 0 \\ Y & Z \end{bmatrix},$$

$\tilde{Q}(\ker(\tilde{U} - \mu I_N)) \subseteq \ker(\phi'(0)^t - \mu I_n)$, so we can just count the dimensions to see if equality holds.

Theorem 6.3 (Main Theorem). *Let ϕ be an analytic self-map of \mathbb{B}^n , fixing 0, not unitary on a slice, and with $\phi'(0)$ full rank. We know that C_ϕ is compact on the Bergman space $A_G^2(\mathbb{B}^n)$, for appropriate G . Let \tilde{Q} be the orthogonal projection of $A_G^2(\mathbb{B}^n)$ to $\langle z_1, \dots, z_n \rangle$. Fix $\mu \in \sigma(\phi'(0))$, the spectrum of $\phi'(0)$, and let*

$$C = \begin{bmatrix} \tilde{U} & 0 \\ \tilde{V} & \tilde{W} \end{bmatrix}$$

with \tilde{U} the compression of C_ϕ to $\langle z^\alpha : |\alpha| \leq K \rangle$, where K is large enough that $\mu \notin \sigma(\tilde{W})$. Now, there is an analytic F_μ with full rank near 0 satisfying Equation (10) if and only if $\tilde{Q}(\ker(\tilde{U} - \mu I_N)) = \ker(\phi'(0)^t - \mu I_n)$, if and only if $\dim[\tilde{Q}(\ker(\tilde{U} - \mu I_N))] = \dim[\ker(\phi'(0)^t - \mu I_n)]$. Finally, there is a full rank solution F satisfying $F \circ \phi = \phi'(0)F$ if and only if $\dim[\tilde{Q}(\ker(\tilde{U} - \mu I_N))] = \dim[\ker(\phi'(0)^t - \mu I_n)]$ for each eigenvalue μ of $\phi'(0)$.

As is the practice in [CM03], it is possible to state this main result without appealing to the underlying Hilbert space or even compact operators, and perhaps a reader with less interest in the operator theory involved and more interest on the functionality of the theorem will appreciate the formulation provided below. Furthermore, we point the interested reader to compare and contrast Theorem 6.4 with [CM03, Theorem 14]. In both theorems it is possibly easiest to consider C_ϕ only as a large (infinite) matrix.

Theorem 6.4 (Main Theorem Revisited). *Let ϕ be an analytic self-map of \mathbb{B}^n , fixing 0, not unitary on a slice, and with $\phi'(0)$ full rank. Write $\sigma(\phi'(0)) = \{\lambda_1, \dots, \lambda_m\}$. Let $N = \max\{\sum k_i : \lambda_1^{k_1} \cdots \lambda_m^{k_m} \in \sigma(\phi'(0))\}$ and write*

$$C_\phi = \begin{bmatrix} U & 0 \\ V & W \end{bmatrix}$$

with U the upper left $N \times N$ corner of C_ϕ . Let $Q : \mathbb{C}^N \rightarrow \mathbb{C}^n$ be the projection to the first n components. A full rank solution to Schroeder's equation exists if and only if $\dim[\ker(Q(U - \lambda_j I_N))] = \dim[\ker(\phi'(0)^t - \lambda_j I_n)]$ for every $j = 1, \dots, m$.

Proof. It suffices to show that $\sigma(\phi'(0)) \cap \sigma(W) = \emptyset$, as the rest follows from 6.3. Choose D to be a unitary $n \times n$ matrix so that $D\phi'(0)D^{-1}$ is upper triangular. Note that $D\mathbb{B}^n = \mathbb{B}^n$ since D is a unitary. We know $C_{D\phi D^{-1}}$ is lower triangular and compact on $A_G^2(\mathbb{B}^n)$, and that C_D is a unitary on $A_G^2(\mathbb{B}^n)$. Now by the choice of N we see that no λ_j can appear on the diagonal of $C_{D\phi D^{-1}}$ below the N^{th} row. Since $C_{D\phi D^{-1}} = C_D^{-1}C_\phi C_D$ it follows that $\sigma(\phi'(0)) \cap \sigma(W) = \emptyset$. \square

While Jordan form of both $\phi'(0)^t$ and U played a large role in understanding and proving the our main results, none of the statements of these results rely on any special matrix form. The main theorem of the section does not rely on any Jordan form, but is instead formulated in terms of invariant subspaces of our operator. This is important for practicality of the theorem; that is, given a specific ϕ , one can theoretically obtain the result of the theorem without first putting $\phi'(0)^t$ into Jordan form.

The following are a few results that can expedite our understanding of when a full rank solutions exists. For the remainder of the section, let C_ϕ and ϕ be as in the hypotheses of Theorem 6.3. Theorem 6.5 is a small generalization of Theorem 5.3.

Theorem 6.5. *If μ is not a resonant eigenvalue, then there exists an F_μ satisfying Equation (10) with full rank near 0.*

Proof. Without loss of generality, we assume $\phi'(0)$ is an upper triangular Jordan form matrix. Since μ does not occur on the the diagonal of C_ϕ below $\phi'(0)^t$, it follows from the proof of Theorem 4.9 that no Jordan block of $\phi'(0)^t$, J_j , can gain length on any induction step of the proof. By Theorem 4.11 we see $Q(\ker(U - \mu I_N)) = Q(\phi'(0)^t - \mu I_n)$, so the result follows from our main theorem. \square

Theorem 6.6. *If μ is a resonant eigenvalue, and $\dim(\ker(U - \mu I_N)) = \dim(\ker(\phi'(0)^t - \mu I_n))$, then there is not a full rank solution F_μ .*

The suggestion of using a counting argument was proposed by Carl C. Cowen, for which the author is thankful.

Proof. Without loss of generality, we assume $\phi'(0)$ is an upper triangular Jordan form matrix. Since μ appears on the diagonal of U below $\phi'(0)^t$ and $\dim(\ker(U - \mu I_N)) = \dim(\ker(\phi'(0)^t - \mu I_n))$, it follows from the proof of Theorem 4.9 that an original Jordan block gains length at some step of the induction. By Theorem 4.11 we see $Q(\ker(U - \mu I_N)) \subsetneq \ker(\phi'(0)^t - \mu I_n)$, and the result follows from Theorem 6.2. \square

Example 6.7. Let $\phi(z) = (\frac{z_1}{2}, \frac{z_2}{4} + \frac{z_3}{8} + \frac{z_1^2}{8}, \frac{z_3}{4}, \frac{z_4}{8})$ on \mathbb{B}^4 . Clearly $|\phi(z)|^2 \leq \frac{|z_1|^2}{4} + \frac{9|z_2|^2}{16} + \frac{9|z_3|^2}{64} + \frac{9|z_1|^4}{64} + \frac{|z_3|^2}{16} + \frac{|z_4|^2}{64} \leq \frac{41|z|^2}{64}$ so ϕ is not unitary on any slice, and fixes 0. Now we have

$$\phi'(0)^t = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 1/8 & 1/4 & 0 \\ 0 & 0 & 0 & 1/8 \end{bmatrix}$$

and

$$C_\phi = \begin{array}{c} \begin{matrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_1^2 \\ z_1 z_2 \\ z_1 z_3 \\ \vdots \end{matrix} \end{array} \begin{bmatrix} \begin{matrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_1^2 & \phi_1 \phi_2 & \phi_1 \phi_3 & \dots \end{matrix} \\ \begin{matrix} 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1/4 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1/8 & 1/4 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1/8 & 0 & 0 & 0 & \dots \\ 0 & 1/8 & 0 & 0 & 1/4 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1/8 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1/16 & 1/16 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix} \end{bmatrix}.$$

By inspection we see that $F_{1/2} = z_1$, and $F_{1/8} = z_4$. So the existence of a full rank solution rests on having a full rank $F_{1/4}$. U can be chosen to be the upper left 6×6 , as all diagonal values below this are less than or equal to $1/16$. Since $\ker(\phi'(0)^t - I_4/4) = \{e_3\} \subset \{e_3, e_5\} = \ker(U - I_6/4)$, we know a full rank solution exists. Now, to find this solution notice that when in Jordan form, $\phi'(0)^t$ has only one $(1/4)$ -Jordan block of length 2, and $e_2, e_3/8$ gives the chain basis of this block. This implies that $F_{1/4} = (f_1, f_2)$ with f_1, f_2 a chain of C_ϕ that is determined by the corresponding chain of U (see Theorem 4.4). Since $(U - I_6/4)e_2 = e_3/8 + e_5/8$, we have $F_{1/4} = (z_2, z_3/8 + z_1^2/8)^t$. In total $F = (z_1, z_2, z_3/8 + z_1^2/8, z_4)$.

7. SOLUTIONS TO $C_\phi F = \phi(0)^k F$

It is well known in one variable that if $\phi : \mathbb{D} \rightarrow \mathbb{D}$ with $\phi(0) = 0$ and $0 < |\phi'(0)| < 1$ then $\phi'(0)^k$ are eigenvectors of C_ϕ for each non-negative integer k . Moreover, given $f \neq 0$ so that $C_\phi f = \phi'(0)f$, it is easy to see that $C_\phi f^k = \phi'(0)^k f$ for $k \geq 1$. Whether or not this phenomenon generalizes to several variables is evidently a natural question to ask in the current circumstances, as the author was asked such a question by Željko Čučković after a presentation at the 27th Southeast Analysis Meetings, University of Florida, Gainesville, March 17-19, 2011. The

author thanks Professor Ćučković for the insightful inquiry, and has included the current section as a belated answer.

Lemma 7.1. *Let J be a lower triangular λ -Jordan block with basis vectors e_1, \dots, e_s . Then for any positive integer k there is a basis $\epsilon_1, \dots, \epsilon_s$ so that $\epsilon_s = e_s$, and when expressed in this basis J^k is a lower triangular λ^k -Jordan block.*

Proof. Working in the original basis, $\{e_1, \dots, e_s\}$ J^k is the lower triangular matrix,

$$J^k = \begin{matrix} e_1 \\ e_2 \\ \vdots \\ e_s \end{matrix} \begin{bmatrix} \lambda^k & & & \\ k\lambda & \lambda^k & & \\ * & \ddots & \ddots & \\ * & * & k\lambda & \lambda^k \end{bmatrix}.$$

Seeing our result is trivial for $s = 1$, we induct on s . Applying our induction hypothesis to the upper left $(s-1) \times (s-1)$ we are furnished with a new basis, namely $\epsilon_1, \dots, \epsilon_{s-2}, e_{s-1}, e_s$, and we have

$$J^k = \begin{matrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ e_{s-1} \\ e_s \end{matrix} \begin{bmatrix} \lambda^k & & & & \\ 1 & \lambda^k & & & \\ & \ddots & \ddots & & \\ & & 1 & \lambda^k & \\ a_1 & \dots & a_{s-2} & k\lambda & \lambda^k \end{bmatrix}.$$

An application of Lemma 4.7 gives our result. \square

Theorem 7.2 (Main theorem of Section 7). *Let ϕ be an analytic self-map of \mathbb{B}^n so that $\phi(0) = 0$, $\phi'(0)$ has full rank, and ϕ is not unitary on any slice. Given a positive integer k , there is an analytic $F : \mathbb{B}^n \rightarrow \mathbb{C}^n$ with linearly independent component functions satisfying*

$$C_\phi F = \phi'(0)^k F.$$

Furthermore, if $k > 1$ then no such F can have full rank near 0.

Proof. To realize the Jordan form of $\phi'(0)$, let D be a nonsingular matrix so that $D\phi'(0)D^{-1} = \text{diag}(J_1^t, \dots, J_m^t)$ with each J_j a lower triangular Jordan block (so J_j^t are the upper triangular counterparts). Say J_j has eigenvalue λ_j and length n_j as usual. Similarly, let E be a nonsingular matrix so that $E\phi'(0)^k E^{-1}$ is an upper triangular Jordan form matrix. It follows from Lemma 7.1 that $E\phi'(0)^k E^{-1}$ is identical to $D\phi'(0)D^{-1}$ except the diagonal entries, λ_j , are replaced with λ_j^k .

Since $(E\phi'(0)E^{-1})^k = E\phi'(0)^k E^{-1}$, $C_{E\phi E^{-1}} F = (E\phi'(0)E^{-1})^k F$ if and only if $C_\phi(E^{-1}FE) = \phi'(0)^k(E^{-1}FE)$. Ergo, it suffices to find an

F which satisfies $C_{E\phi E^{-1}}F = (E\phi'(0)E^{-1})^k F$, and has linearly independent component functions. Mimicking Lemma 3.2, we see that such an F exists if and only if for each $1 \leq j \leq m$, C_ϕ has a chain of length n_j and eigenvalue λ_j^k and that the chains form a linearly independent set.

Set $\psi = D\phi D^{-1}$. It follows from the proof of Proposition 2.1 that $C_\phi = \iota^{-1}C_\psi\iota$ with ι an isometric-isomorphism. Consequently, it suffices to show that the corresponding chains exist for C_ψ .

For notational simplicity, let $\lambda_1 = \lambda$, and set $n_1 = s$. Now, since $\psi'(0)$ is in Jordan form, we see $\psi_j(z) = \lambda z_j + z_{j+1} + O(2)$ for $1 \leq j < s$, and $\psi_s(z) = \lambda z_s + O(2)$. We calculate,

$$\begin{aligned} \psi_1^k &= [\lambda z_1 + z_2 + O(2)]^k \\ &= \lambda^k z_1^k + k\lambda^{k-1}z_1^{k-1}z_2 + \cdots + z_2^k + O(k+1) \\ \psi_1^{k-1}\psi_j &= [\lambda z_1 + z_2 + O(2)]^{k-1}[\lambda z_j + z_{j+1} + O(2)] \\ &= [\lambda^{k-1}z_1^{k-1} + (k-1)\lambda^{k-2}z_1^{k-2}z_2 + \cdots + \\ &\quad z_2^{k-1} + O(k)][\lambda z_j + z_{j+1} + O(2)] \\ &= \lambda^k z_1^{k-1}z_j + \lambda^{k-1}z_1^{k-1}z_{j+1} + \cdots + z_2^{k-1}z_{j+1} + O(k+1). \end{aligned}$$

Hence, C_ψ contains the following sub-matrix along it's diagonal,

$$\begin{array}{c} z_1^k \\ z_1^{k-1}z_2 \\ \vdots \\ z_1^{k-1}z_s \end{array} \begin{bmatrix} \phi_1^k & \phi_1^{k-1}\phi_2 & \cdots & \phi_1^{k-1}\phi_s \\ \lambda^k & 0 & 0 & 0 \\ \lambda^{k-1} & \lambda^k & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \lambda^{k-1} & \lambda_1^k \end{bmatrix}.$$

Similarly, we have such a sub-matrix of C_ψ for each λ_j along the diagonal of C_ψ . It follows from Theorem 4.3 and Theorem 4.12 that for each $1 \leq j \leq m$, C_ψ has a chain of length n_j and eigenvalue λ_j , and that these chains are linearly independent.

We have proven that $C_\phi F = \phi'(0)^k F$ can be solved with an F having linearly independent component functions. It remains to show that any such F must have $F'(0)$ singular if $k \geq 2$. Let $k \geq 2$, $A = F'(0)$, and $B = \phi'(0)$. Using the notation above we know the spectrum of B is $\sigma(B) = \{\lambda_1, \dots, \lambda_m\}$ and $\sigma(B^k) = \{\lambda_1^k, \dots, \lambda_m^k\}$. With no loss of generality, set $\lambda = \lambda_1$, and assume $|\lambda| = \max \sigma(B)$. Differentiating $F \circ \phi = \phi'(0)^k F$ and evaluating at $z = 0$ gives $AB = B^k A$. Hence, if we let e be a nonzero member of $\ker(B - \lambda I_n)$ we obtain

$$B^k(Ae) = ABe = \lambda Ae.$$

Since $k \geq 2$, $|\lambda| > |\lambda|^k = \max \sigma(B^k)$. It follows that λ is not within $\sigma(B)$, and thus, $Ae = 0$, which concludes the proof. \square

The existence proof above gives little insight on how to find such a solution in practice. Driven by the single variable setting, in which $C_\phi f = \phi'(0)f$ implies $C_\phi f^k = \phi'(0)^k$ for any positive integer k , we have the following useful result.

Theorem 7.3. *Suppose $F = (f_1, \dots, f_s)^t$ has linearly independent component functions and satisfies $C_\phi F = J^t F$ with*

$$J^t = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}$$

an upper triangular $s \times s$ Jordan block, and $\lambda \neq 0$. For an integer $k \geq 1$, let E be the $s \times s$ invertible matrix provided by Lemma 7.1 so that

$$E(J^t)^k E^{-1} = \begin{bmatrix} \lambda^k & 1 & & \\ & \ddots & \ddots & \\ & & \lambda^k & 1 \\ & & & \lambda^k \end{bmatrix}.$$

Then

$$G = E^{-1} \left(\frac{f_1 f_s^{k-1}}{\lambda^{(k-1)(s-1)}}, \dots, \frac{f_{s-1} f_s^{k-1}}{\lambda^{(k-1)(1)}}, \frac{f_s^k}{\lambda^{(k-1)(0)}} \right)^t$$

satisfies $C_\phi G = (J^t)^k G$ and has linearly independent component functions.

Proof. $C_\phi F = J^t F$ is equivalently written as

$$(C_\phi - \lambda I) : f_1 \mapsto f_2 \mapsto \dots \mapsto f_s \mapsto 0.$$

For $j = 1, \dots, s$ set

$$h_j = \frac{f_j f_s^{k-1}}{\lambda^{(k-1)(s-j)}},$$

and $H = (h_1, \dots, h_s)^t$. A simple calculation yields

$$C_\phi h_s = C_\phi f_s^k = \lambda^k f_s^k = \lambda^k h_s,$$

and for $1 \leq j < s$

$$\begin{aligned}
C_\phi h_j &= C_\phi \frac{f_j f_s^{k-1}}{\lambda^{(k-1)(s-j)}} \\
&= \frac{(\lambda f_j + f_{j+1}) \lambda^{k-1} f_s^{(k-1)}}{\lambda^{(k-1)(s-j)}} \\
&= \lambda^k \frac{f_j f_s^{(k-1)}}{\lambda^{(k-1)(s-j)}} + \frac{f_{j+1} f_s^{(k-1)}}{\lambda^{(k-1)(s-j-1)}} \\
&= \lambda^k h_j + h_{j+1}.
\end{aligned}$$

Equivalently,

$$(C_\phi - \lambda^k I) : h_1 \mapsto h_2 \mapsto \dots \mapsto h_s \mapsto 0.$$

So we have

$$C_\phi H = \begin{bmatrix} \lambda^k & 1 & & \\ & \ddots & \ddots & \\ & & \lambda^k & 1 \\ & & & \lambda^k \end{bmatrix} H = (E(J^t)^k E^{-1}) H.$$

It follows that $G := E^{-1}H$ satisfies $C_\phi G = (J^t)^k G$ as desired.

Lastly, note that $\{h_j\}_j = f_s^{(k-1)} \{c_1 f_1, \dots, c_s f_s\}$ with $c_j = \lambda^{-(k-1)(s-j)}$, and so is a linearly independent set. Now for any non-zero $a \in \mathbb{C}^s$ we have $0 \neq \langle H, a \rangle = \langle EG, a \rangle = \langle G, E^* a \rangle$. Since E^* is bijective, this implies that G has linearly independent component functions. \square

Note that if $\phi'(0)$ has full rank, then none of its eigenvalues can be zero, so the hypothesis $\lambda \neq 0$ in the theorem above is not restrictive. Sections 2 and 4 assure us we can find a full rank solution to Schroeder's equation, and allow us to first put $\phi'(0)$ into Jordan form, and then find such a solution (by finding the chains of U and subsequently C_ϕ). Thus, Theorem 7.3 now allows us to explicitly find solutions to $C_\phi F = \phi'(0)^k F$ that have linearly independent component functions. If $k > 1$, Theorem 7.2 says that no full rank solution exists, so in some sense this is the best we can do. The case $k = 1$ of course is handled in Section 6.

The following corollary is known, for example [CM95, Theorem 7.20], but since it follows immediately, we state it for completeness.

Corollary 7.4. *Let ϕ be a self-map of \mathbb{B}^n fixing 0, and suppose ϕ is not unitary on any slice, and $\phi'(0)$ has full rank. Then for all $\lambda \in \sigma(\phi'(0))$, and for all non-negative integers k , λ^k is an eigenvalue of C_ϕ .*

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